Minimally Intersecting Set Partitions of Type B

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Abstract

Motivated by Pittel's study of minimally intersecting set partitions, we investigate minimally intersecting set partitions of type B. We find a formula for the number of minimally intersecting r-tuples of B_n-partitions, as well as a formula for the number of minimally intersecting r-tuples of B_n-partitions without zero-block. As a consequence, it follows the formula of Benoumhani for the Dowling number in analogy to Dobiński's formula.

Keywords: minimally intersecting B_n-partitions, Dobiński's formula, the Dowling number

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1 Introduction

This paper is primarily concerned with the meet structure of the lattice of type B_n partitions of the set $[\pm n] = \{\pm 1, \pm 2, \ldots, \pm n\}$, as well as of the meet-semilattice of type B_n partitions without zero-block. The lattice structure of type B_n set partitions has been studied by Reiner [8]. It can be regarded as a representation of the intersection lattice of the type B Coxeter arrangements, see Björner and Wachs [3], Björner and Brenti [2] and Humphreys [6].

We establish a formula for the number of B_n-partitions π' which minimally intersect a given B_n-partition π. Using the same technique, we derive a formula for the number of B_n-partitions π' without zero-block which minimally intersect a given B_n-partition π without zero-block. The ordinary case has been studied by Pittel [7]. In particular, if we take π to be the minimal B_n-partition, our formula reduces to a formula of Benoumhani [1] for the number of B_n-partitions (called the Dowling number, see Dowling [5]), which is analogous to Dobiński's formula for the number of partitions of a finite set.

In a more general setting, we derive two formulas for the number of minimally intersecting r-tuples of B_n-partitions and the number of minimally intersecting r-tuples of B_n-partitions without zero-block. Recall that Canfield [4] has found a relation

between the exponential generating function of the number of minimally intersecting r-tuples of partitions and the powers of the Bell numbers. We give a type B analogue of this relation.

Let us give an overview of relevant notation and terminology. A partition of a set S is a collection $\{B_1, B_2, \ldots, B_k\}$ of subsets of S such that $B_1 \cup B_2 \cup \cdots \cup B_k = S$ and $B_i \cap B_j = \emptyset$ for any $i \neq j$. A *set partition of type* B_n is a partition π of the set $[\pm n]$ into blocks satisfying the following conditions:

(i) For any block B of π, its opposite $-B$ obtained by negating all elements of B is also a block of π;

(ii) There is at most one *zero-block*, which is defined to be a block B such that $B = -B$.

We call $\pm B$ a *block pair* of π if B is a non-zero-block of π. For example,

$$\pi_1 = \{\{\pm 1,\ \pm 2,\ \pm 5,\ \pm 8,\ \pm 12\},\ \pm\{3,\ 11\},\ \pm\{4,\ -7,\ 9,\ 10\},\ \pm\{6\}\}$$

is a B_{12}-partition consisting of 3 block pairs and the zero-block $\{\pm 1,\ \pm 2,\ \pm 5,\ \pm 8,\ \pm 12\}$.

The total number of partitions of the set $[n] = \{1, 2, \ldots, n\}$ is called the Bell number and is denoted by B_n, see Rota [11]. The type B analogue of the Bell numbers are the Dowling numbers $\left|\Pi_n^B\right|$, where Π_n^B denotes the set of B_n-partitions. The sequence $\{|\Pi_n^B|\}_{n\geq 0}$ is listed as A007405 in [12]:

$$1,\ 2,\ 6,\ 24,\ 116,\ 648,\ 4088,\ 28640,\ 219920,\ 1832224,\ \ldots.$$

Let π and π' be two partitions of the set $[n]$. We say that π *refines* π' if every block of π is contained in some block of π'. The refinement relation is a partial ordering of the set Π_n of all partitions of $[n]$. Define the *meet*, denoted $\pi \wedge \pi'$, to be the largest partition which refines both π and π'. Define their *join*, denoted $\pi \vee \pi'$, to be the smallest partition which is refined by both π and π'. The poset Π_n is a lattice with the minimum element $\hat{0} = \{\{1\}, \{2\}, \ldots, \{n\}\}$. We say that the partitions $\pi_1, \pi_2, \ldots, \pi_r$ intersect minimally if $\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_r = \hat{0}$.

Pittel [7] has found a formula for the number of partitions minimally intersecting a given partition. He also computed the number of minimally intersecting r-tuples of partitions.

Theorem 1.1. *Let π be a partition of $[n]$, and let $i_1, \ldots, i_k$ be the sizes of the blocks of π listed in any order. Then the number of partitions intersecting π minimally equals*

$$N(\pi) = \mathbf{i}!\,\left[\mathbf{x}^{\mathbf{i}}\right] \exp\left(\prod_{\alpha\in[k]} (1 + x_\alpha) - 1\right),$$

where $\mathbf{i}! = \prod_{\alpha\in[k]} i_\alpha!$ *and* $\left[\mathbf{x}^{\mathbf{i}}\right]$ *stands for the coefficient of* $\mathbf{x}^{\mathbf{i}} = \prod_{\alpha\in[k]} x_\alpha^{i_\alpha}$ *of a power series in* $x_1, x_2, \ldots, x_k$*. Let* $r \geq 2$*. The number* $N_{n,r}$ *of minimally intersecting* r*-tuples* $(\pi_1, \pi_2, \ldots, \pi_r)$ *of partitions is given by*

$$N_{n,r} = \frac{1}{e^r} \sum_{k_1,\ldots,k_r \geq 0} \frac{(k_1 k_2 \cdots k_r)_n}{k_1! k_2! \cdots k_r!},$$

where the notation $(x)_n = x(x-1)\cdots(x-n+1)$ *denotes the falling factorial.*

By taking $\pi = \hat{0}$, the above formula reduces to Dobiński's formula

$$B_n = \frac{1}{e} \sum_{k \geq 0} \frac{k^n}{k!}, \tag{1.1}$$

see Rota [11]. Wilf has obtained the following alternative formula

$$N_{n,r} = \sum_{j=1}^{n} B_j^r s(n,j), \tag{1.2}$$

where $s(n,j)$ is the Stirling number of the first kind. Denote the generating function of $N_{n,r}$ by

$$M_r(x) = \sum_{n \geq 0} N_{n,r} \frac{x^n}{n!}.$$

Canfield [4] has established the following connection between $M_r(x)$ and the Bell numbers:

$$M_r\left(e^x - 1\right) = \sum_{n \geq 0} B_n^r \frac{x^n}{n!}. \tag{1.3}$$

We shall give type B analogues of (1.2) and (1.3) based on type B partitions without zero block.

This paper is organized as follows. In Section 2, we give an expression for the number of B_n-partitions that minimally intersect a B_n-partition π of a given type, which contains Benoumhani's formula for the Dowling number as a special case. Moreover, we obtain a formula for the number of minimally intersecting r-tuples of B_n-partitions. In Section 3, we consider the enumeration of minimally intersecting r-tuples of B_n-partitions without zero-block, and give two formulas in analogy to (1.2), and (1.3).

2 Minimally intersecting B_n-partitions

The main objective of this section is to derive a formula for the number of minimally intersecting r-tuples of B_n-partitions. If $\pi \in \Pi_n^B$ has a zero-block $Z = \{\pm i_1, \pm i_2, \ldots, \pm i_k\}$, we say that Z is of *half-size* k. The partition $\hat{0}^B = \{\{1\}, \{-1\}, \{2\}, \{-2\}, \ldots, \{n\}, \{-n\}\}$

is called the minimal partition, and $\hat{1}^B = \{\{\pm 1, \pm 2, \ldots, \pm n\}\}$ is called the maximal partition. We say that $\pi_1, \pi_2, \ldots, \pi_r$ are *minimally intersecting* if $\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_r = \hat{0}^B$.

Let $\mathbf{j} = (j_1, j_2, \ldots, j_k)$ be a composition of n. Let π be a B_n-partition consisting of k block pairs and a zero-block of half-size i_0. For the purpose of enumeration, we often assume that the block pairs of π are ordered subject to certain convention. We say that π is of *type* $(i_0; \mathbf{j})$ if the block pairs of π are ordered such that the i-th block pair is of length j_i.

We first consider the problem of counting the number of B_n-partitions with l block pairs which minimally intersects a given B_n-partition. As a special case, we are led to Benoumhani's formula for the Dowling number

$$\left|\Pi_n^B\right| = \frac{1}{\sqrt{e}} \sum_{k \geq 0} \frac{(2k+1)^n}{(2k)!!}, \tag{2.1}$$

in analogy to Dobiński's formula (1.1). Next, we find a formula for the number of ordered pairs of minimally intersecting B_n-partitions. In general, we give a formula for the number of minimally intersecting r-tuples of B_n-partitions.

Theorem 2.1. *Let π be a B_n-partition consisting of a zero-block of half-size i_0 (allowing $i_0 = 0$) and k block pairs of sizes $i_1, i_2, \ldots, i_k$ ($k \geq 1$) listed in any order. For any $l \geq 1$, the number of B_n-partitions π' containing exactly l block pairs that intersect π minimally equals*

$$N^B(\pi; l) = \frac{\mathbf{i}!}{(2l - 2i_0)!!} \sum_{\mathbf{i}'} \left[\mathbf{x}^{\mathbf{i}'}\right] \left(\prod_{\alpha \in [k]} (1 + x_\alpha)^2 - 1 \right)^{l - i_0} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2i_0}, \tag{2.2}$$

where $\mathbf{i}'$ runs over all vectors $(i'_1, i'_2, \ldots, i'_k)$ such that $i'_\alpha \in \{i_\alpha, i_\alpha - 1\}$ for any $\alpha \in [k]$, and $\mathbf{x}^{\mathbf{i}'} = \prod_{\alpha=1}^{k} x_\alpha^{i'_\alpha}$.

For example, Π_2^B contains 6 partitions:

$$\hat{0}^B,\ \hat{1}^B,\ \{\pm\{1\}, \{\pm 2\}\},\ \{\pm\{2\}, \{\pm 1\}\},\ \{\pm\{1, 2\}\},\ \{\pm\{1, -2\}\}.$$

Let $\pi = \{\pm\{1\}, \{\pm 2\}\}$. We have $i_0 = 1$, $k = 1$, and $i_1 = 1$. For $l = 1$, by (2.2), $N^B(\pi; 1) = \sum_{i=0}^{1} [x^i] (1 + x)^2 = 3$. The three B_2-partitions which contain exactly 1 block pair and intersect π minimally are $\{\pm\{2\}, \{\pm 1\}\}$, $\{\pm\{1, 2\}\}$, and $\{\pm\{1, -2\}\}$.

Proof of Theorem 2.1. Let Z_1 be the zero-block of π, and $\pm B_1, \pm B_2, \ldots, \pm B_k$ be the block pairs of π. Let Z_2 be the zero-block of π', and $\pm B'_1, \pm B'_2, \ldots, \pm B'_l$ be the block pairs of π'. To ensure that π and π' are minimally intersected, it is necessary to characterize the intersecting relations for all pairs (B, B') where B is a block of π and B' is a block of π'.

First, we observe that the intersection $B \cap B'$ contains at most one element subject to the minimally intersecting property. In particular, $Z_1 \cap Z_2 = \emptyset$. If $B = Z_1$ and

$B' \neq Z_2$, then the two intersections $Z_1 \cap B'$ and $Z_1 \cap (-B')$ are a pair of opposite subsets. This observation allows us to disregard $Z_1 \cap (-B')$ in our consideration. Since the cardinality of $B \cap B'$ is either zero or one, we can represent $B \cap B'$ by

$$F(k;l) \prod_{\beta \in [l]} (1 + z_1 w_\beta) \prod_{\alpha \in [k]} (1 + x_\alpha z_2),$$

where

$$F(k;l) = \prod_{\alpha \in [k],\, \beta \in [l]} (1 + x_\alpha y_\beta)(1 + x_\alpha \bar{y}_\beta). \tag{2.3}$$

Here we use x_i (w_i, resp.) to represent one of the two blocks in the i-th block pair of π (π', resp.), and we use y_i and $\bar{y}_i$ to represent the two blocks in the i-th block pair of π'.

The above argument allows us to generate all B_n-partitions that minimally meet with π. Let us consider the generating function of such B_n-partitions. Set

$$\begin{array}{lll}
\mathbf{x} = (x_1, x_2, \ldots, x_k), & \mathbf{i} = (i_1, i_2, \ldots, i_k), & \mathbf{x}^{\mathbf{i}} = \prod_{\alpha \in [k]} x_\alpha^{i_\alpha}; \\
\mathbf{w} = (w_1, w_2, \ldots, w_l), & \mathbf{a} = (a_1, a_2, \ldots, a_l), & \mathbf{w}^{\mathbf{a}} = \prod_{\beta \in [l]} w_\beta^{a_\beta}; \\
\mathbf{y} = (y_1, y_2, \ldots, y_l), & \mathbf{b} = (b_1, b_2, \ldots, b_l), & \mathbf{y}^{\mathbf{b}} = \prod_{\beta \in [l]} y_\beta^{b_\beta}; \\
\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \ldots, \bar{y}_l), & \mathbf{c} = (c_1, c_2, \ldots, c_l), & \bar{\mathbf{y}}^{\mathbf{c}} = \prod_{\beta \in [l]} \bar{y}_\beta^{c_\beta}.
\end{array}$$

Let j_0 be a nonnegative integer and $\mathbf{j} = (j_1, j_2, \ldots, j_l)$ a composition of $n - j_0$. Denote by $N^B(\pi; j_0, \mathbf{j})$ the number of B_n-partitions π' of type $(j_0; \mathbf{j})$ such that π' meets π minimally. In the above notation, we have

$$N^B(\pi; j_0, \mathbf{j}) = c \cdot \sum_{\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{j}} \left[\mathbf{x}^{\mathbf{i}} z_1^{i_0} z_2^{j_0} \mathbf{w}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}} \bar{\mathbf{y}}^{\mathbf{c}}\right] F(k;l) \prod_{\beta \in [l]} (1 + z_1 w_\beta) \prod_{\alpha \in [k]} (1 + x_\alpha z_2), \tag{2.4}$$

where $c = \mathbf{i}! \cdot (2i_0)!!/(2l)!!$. Denote by $\binom{S}{m}$ the collection of all m-subsets of S. Since

$$\left[z_1^{i_0}\right] \prod_{\beta \in [l]} (1 + z_1 w_\beta) = \sum_{Y \in \binom{[l]}{i_0}} \prod_{\beta \in Y} w_\beta, \tag{2.5}$$

$$\left[z_2^{j_0}\right] \prod_{\alpha \in [k]} (1 + x_\alpha z_2) = \sum_{X \in \binom{[k]}{j_0}} \prod_{\alpha \in X} x_\alpha, \tag{2.6}$$

substituting (2.5) and (2.6) into (2.4), we obtain that

$$\begin{aligned}
N^B(\pi; j_0, \mathbf{j}) &= c \cdot \sum_{\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{j}} \left[\mathbf{x}^{\mathbf{i}} \mathbf{w}^{\mathbf{a}} \mathbf{y}^{\mathbf{b}} \bar{\mathbf{y}}^{\mathbf{c}}\right] \left(\sum_{Y \in \binom{[l]}{i_0}} \prod_{\beta \in Y} w_\beta \right) \left(\sum_{X \in \binom{[k]}{j_0}} \prod_{\alpha \in X} x_\alpha \right) F(k;l) \\
&= c \cdot \sum_{X, Y, \mathbf{b}} \left[\mathbf{y}^{\mathbf{b}} \prod_{\alpha \in [k]} x_\alpha^{i_\alpha - \chi(\alpha \in X)} \prod_{\beta \in [l]} \bar{y}_\beta^{j_\beta - b_\beta - \chi(\beta \in Y)} \right] F(k;l),
\end{aligned}$$

where χ is the characteristic function defined by $\chi(P) = 1$ if P is true, and $\chi(P) = 0$ otherwise. Therefore

$$N^B(\pi; l) = \sum_{\substack{j_0+j_1+\cdots+j_l=n \\ j_0 \geq 0,\, j_1,\ldots,j_l \geq 1}} N^B(\pi; j_0, \mathbf{j}) = c \cdot \sum_{j_0,\, X} \left[\prod_{\alpha} x_\alpha^{i_\alpha - \chi(\alpha \in X)} \right] \sum_{\substack{j_0+j_1+\cdots+j_l=n \\ j_1,\ldots,j_l \geq 1}} f(\mathbf{j}), \quad (2.7)$$

where

$$f(\mathbf{j}) = \sum_{Y, \mathbf{b}} \left[\mathbf{y}^{\mathbf{b}} \prod_{\beta} \bar{y}_\beta^{j_\beta - b_\beta - \chi(\beta \in Y)} \right] F(k; l).$$

In view of the expression (2.3), the total degree of x_α's agrees with the sum of the total degrees of y_β's and $\bar{y}_\beta$'s in $F(k; l)$. In other words,

$$\sum_{\alpha \in [k]} i_\alpha - \chi(\alpha \in X) = \sum_{\beta \in [l]} b_\beta + (j_\beta - b_\beta - \chi(\beta \in Y)),$$

namely, $j_0 + j_1 + \cdots + j_l = n$. So we may drop this condition in the inner summation of (2.7). For any $A \subseteq [l]$, let

$$S(A) = \sum_{\substack{j_1,\ldots,j_l \geq 0 \\ j_\beta = 0 \text{ if } \beta \notin A}} f(\mathbf{j}) = \sum_{Y} \sum_{\substack{b_\gamma, j_\gamma \geq 0 \\ \gamma \in A}} \left[\prod_{\gamma \in A} y_\gamma^{b_\gamma} \bar{y}_\gamma^{j_\gamma - b_\gamma - \chi(\gamma \in Y)} \right] F(k; A),$$

where

$$F(k; A) = \prod_{\alpha \in [k],\, \gamma \in A} (1 + x_\alpha y_\gamma)(1 + x_\alpha \bar{y}_\gamma).$$

Since j_γ and b_γ run over all nonnegative integers, the exponent $j_\gamma - b_\gamma - \chi(\gamma \in Y)$ can considered as a summation index. It follows that

$$S(A) = \sum_{Y \in \binom{A}{i_0}} \sum_{b_\gamma, c_\gamma \geq 0,\, \gamma \in A} \left[\prod_{\gamma \in A} y_\gamma^{b_\gamma} \bar{y}_\gamma^{c_\gamma} \right] F(k; A) = \binom{|A|}{i_0} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2|A|}.$$

By the principle of inclusion-exclusion, we have

$$\begin{aligned} \sum_{j_1,\ldots,j_l \geq 1} f(\mathbf{j}) &= \sum_{A \subseteq [l]} (-1)^{l-|A|} S(A) = \sum_{m} \binom{l}{m} (-1)^{l-m} \binom{m}{i_0} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2m} \\ &= \binom{l}{i_0} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2i_0} \left(\prod_{\alpha \in [k]} (1 + x_\alpha)^2 - 1 \right)^{l - i_0}. \end{aligned}$$

Now, employing (2.7) we find that $N^B(\pi; l)$ equals

$$\frac{\mathbf{i}!}{(2l - 2i_0)!!} \sum_{X \subseteq [k]} \left[\prod_{\alpha \in [k]} x_\alpha^{i_\alpha - \chi(\alpha \in X)} \right] \prod_{\alpha \in [k]} (1 + x_\alpha)^{2i_0} \left(\prod_{\alpha \in [k]} (1 + x_\alpha)^2 - 1 \right)^{l - i_0}, \quad (2.8)$$

which can be rewritten in the form of (2.2). This completes the proof. ∎

The formula (2.8) will also be used in the proof of Corollary 3.1. Summing (2.2) over $l \geq i_0$, we obtain the following formula.

Corollary 2.2. *The number $N^B(\pi)$ of B_n-partitions that minimally intersect π is*

$$N^B(\pi) = \frac{\mathbf{i}!}{\sqrt{e}} \sum_{\mathbf{i}'} \left[\mathbf{x}^{\mathbf{i}'}\right] F(\mathbf{x}), \tag{2.9}$$

where

$$F(\mathbf{x}) = \left(\prod_{\alpha \in [k]} (1 + x_\alpha)^{2i_0} \right) \exp \left(\frac{1}{2} \prod_{\alpha \in [k]} (1 + x_\alpha)^2 \right). \tag{2.10}$$

Setting $\pi = \hat{0}^B$, (2.9) reduces to (2.1), since

$$N^B(\hat{0}^B) = \frac{1}{\sqrt{e}} \sum_{i'_\alpha \in \{0,1\}} \left[x_1^{i'_1} \cdots x_n^{i'_n}\right] \sum_{j \geq 0} \frac{1}{(2j)!!} \prod_{\alpha=1}^{n} (1 + x_\alpha)^{2j}.$$

In fact, the number $N^B(\pi)$ can be expressed in terms of an infinite sum.

Corollary 2.3.

$$N^B(\pi) = \frac{1}{\sqrt{e}} \sum_{j \geq 0} \frac{(2i_0 + 2j + 1)!^k}{(2j)!!} \prod_{\alpha \in [k]} \frac{1}{(2i_0 + 2j + 1 - i_\alpha)!}. \tag{2.11}$$

Proof. From (2.10) it follows that

$$F(x) = \sum_{j \geq 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} (1 + x_\alpha)^{2(i_0 + j)}.$$

Hence

$$\begin{aligned} N^B(\pi) &= \frac{\mathbf{i}!}{\sqrt{e}} \sum_{j \geq 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} \left(\binom{2(i_0 + j)}{i_\alpha} + \binom{2(i_0 + j)}{i_\alpha - 1} \right) \\ &= \frac{\mathbf{i}!}{\sqrt{e}} \sum_{j \geq 0} \frac{1}{(2j)!!} \prod_{\alpha \in [k]} \binom{2(i_0 + j) + 1}{i_\alpha}, \end{aligned}$$

which gives (2.11). This completes the proof. ∎

Corollary 2.4. *Let $N^B_{n,2}(i_0; k)$ denote the number of ordered pairs (π, π') of minimally intersecting B_n-partitions such that π consists of exactly k block pairs and a zero-block of half-size i_0 (allowing $i_0 = 0$). Then*

$$N^B_{n,2}(i_0; k) = \frac{(2n)!!}{(2i_0)!!(2k)!!\sqrt{e}} \left[x^{n - i_0}\right] \sum_{j \geq 0} \frac{1}{(2j)!!} \left((1 + x)^{2i_0 + 2j + 1} - 1 \right)^k. \tag{2.12}$$

Proof. By a simple combinatorial argument we see that the number of B_n-partitions of type $(i_0; i_1, \ldots, i_k)$ equals

$$c = \binom{n}{i_0, i_1, \ldots, i_k} \frac{2^{n-i_0-k}}{k!} = \frac{(2n)!!}{(2i_0)!!(2k)!!} \cdot \frac{1}{\mathbf{i}!}.$$

Thus by (2.9), we have

$$N_{n,2}^B(k) = \sum_{\substack{i_0+i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 1}} c \cdot N^B(\pi) = \frac{(2n)!!}{(2i_0)!!(2k)!!\sqrt{e}} \sum_{\substack{i_0+i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 1}} \sum_{\mathbf{i}'} \left[\mathbf{x}^{\mathbf{i}'}\right] F(\mathbf{x}). \quad (2.13)$$

For any $A \subseteq [k]$, consider

$$S(A) = \sum_{\substack{i_0+i_1+\cdots+i_k=n \\ i_1,\ldots,i_k \geq 0 \\ i_\alpha = 0 \text{ if } \alpha \notin A}} \sum_{\mathbf{i}'} \left[\mathbf{x}^{\mathbf{i}'}\right] F(\mathbf{x}) = \sum_{\substack{i_0+\sum_{\alpha\in A} i_\alpha = n \\ i_\alpha \geq 0,\, \alpha \in A}} \sum_{\mathbf{i}'|_A} \left[\mathbf{x}^{\mathbf{i}'}\big|_A\right] F\left(\mathbf{x}\big|_A\right),$$

where $\mathbf{x}\big|_A$ (resp. $\mathbf{i}'|_A$) denotes the vector obtained by removing all x_α (resp. i'_α) such that $\alpha \notin A$ from the vector $\mathbf{x}$ (resp. $\mathbf{i}'$). Let t be the number of α's such that $i'_\alpha = i_\alpha - 1$ in the inner summation. Noting that

$$F\left(\mathbf{x}\big|_A\right) = \left(\prod_{\alpha\in A}(1+x_\alpha)^{2i_0}\right) \exp\left(\frac{1}{2}\prod_{\alpha\in A}(1+x_\alpha)^2\right),$$

we can transform $S(A)$ to

$$\begin{aligned} S(A) &= \left(\sum_t \binom{|A|}{t} \left[x^{n-i_0-t}\right]\right) (1+x)^{2i_0|A|} \exp\left(\frac{1}{2}(1+x)^{2|A|}\right) \\ &= \left[x^{n-i_0}\right] (1+x)^{(2i_0+1)|A|} \exp\left(\frac{1}{2}(1+x)^{2|A|}\right). \end{aligned}$$

In view of the principle of inclusion-exclusion, we deduce from (2.13) that

$$N_{n,2}^B(k) = \frac{(2n)!!}{(2i_0)!!(2k)!!\sqrt{e}} \sum_{A\subseteq[k]} (-1)^{k-|A|} S(A),$$

which gives (2.12). This completes the proof. ∎

Summing over $0 \leq k \leq n - i_0$ and $0 \leq i_0 \leq n$, we obtain the number of ordered pairs of minimally intersecting B_n-partitions.

Corollary 2.5. *The number $N_{n,2}^B$ of ordered pairs (π, π') of minimally intersecting B_n-partitions is given by*

$$N_{n,2}^B = \frac{2^n}{e} \sum_{k,l\geq 0} \frac{(2kl+k+l)_n}{(2k)!!(2l)!!}.$$

For example, $N_{1,2}^B = 3$, $N_{2,2}^B = 23$, $N_{3,2}^B = 329$, $N_{4,2} = 6737$. In general, we have the following theorem, which is the main result of this paper.

Theorem 2.6. *Let $r \geq 2$. The number of minimally intersecting r-tuples $(\pi_1, \pi_2, \ldots, \pi_r)$ of B_n-partitions equals*

$$N_{n,r}^B = \frac{2^n}{e^{r/2}} \sum_{l_1,l_2,\ldots,l_r} \frac{(f_r)_n}{(2l_1)!!(2l_2)!! \cdots (2l_r)!!}, \tag{2.14}$$

where

$$f_r = \frac{1}{2}\left(\prod_{t\in[r]} (2l_t + 1) - 1\right).$$

Proof. For any $t \in [r]$, let i_t be an nonnegative integer and $\mathbf{j}_t = (j_{t,1}, j_{t,2}, \ldots, j_{t,k_t})$ be a composition of n. Let π_t be a B_n-partition of type $(i_t; \mathbf{j}_t)$. The condition that $\pi_1, \pi_2, \ldots, \pi_r$ are minimally intersecting leads us to consider the intersecting relations for all r-tuples $(B_1, B_2, \ldots, B_r)$ where B_t is a block of π_t.

First, we observe that the intersection

$$B_1 \cap B_2 \cap \cdots \cap B_r \tag{2.15}$$

contains at most one element because of the minimally intersecting requirement. In particular, (2.15) is empty when $B_1, B_2, \ldots, B_r$ are all zero-blocks. We now consider the case that not all of $B_1, B_2, \ldots, B_r$ are zero-blocks. In this case, there exists a number $t \in [r]$ such that $B_1, \ldots, B_{t-1}$ are zero-blocks but B_t is a non-zero-block. This number t will play a key role in determining the intersection (2.15).

In fact, the partial intersection $B_1 \cap B_2 \cap \cdots \cap B_{t-1}$ is of the form $\{\pm i_1, \ldots, \pm i_j\}$. Thus for any non-zero-block B of π_t, the two intersections

$$B_1 \cap \cdots \cap B_{t-1} \cap B \quad \text{and} \quad B_1 \cap \cdots \cap B_{t-1} \cap (-B)$$

form a pair of opposite subsets. This observation allows us to consider B as a representative of the block pair $\pm B$. Since the cardinality of the intersection (2.15) is either zero or one, we can represent (2.15) by

$$f = 1 + z_1 \cdots z_{t-1} x_{t,\alpha_t} Y_{t+1} \cdots Y_r, \tag{2.16}$$

where

$$Y_p \in \left\{z_p,\ y_{p,1},\ \bar{y}_{p,1},\ \ldots,\ y_{p,k_p},\ \bar{y}_{p,k_p}\right\}$$

for $p \geq t+1$. Here we use z_i to represent the zero-block of π_i, $x_{t,i}$ to represent one of the two blocks in the i-th block pair of π_t, $y_{p,i}$ and $\bar{y}_{p,i}$ to represent the two blocks in

the i-th block pair of π_p. Let

$$\mathbf{x}_t = (x_{t,1}, \ldots, x_{t,k_t}), \qquad \mathbf{a}_t = (a_{t,1}, \ldots, a_{t,k_t}), \qquad \mathbf{x}_s^{\mathbf{a}_s} = \prod_{i\in[k_s]} x_{s,i}^{a_{s,i}};$$

$$\mathbf{y}_t = (y_{t,1}, \ldots, y_{t,k_t}), \qquad \mathbf{b}_t = (b_{t,1}, \ldots, b_{t,k_t}), \qquad \mathbf{y}_s^{\mathbf{b}_s} = \prod_{i\in[k_s]} y_{s,i}^{b_{s,i}};$$

$$\bar{\mathbf{y}}_t = (\bar{y}_{t,1}, \ldots, \bar{y}_{t,k_t}), \qquad \mathbf{c}_t = (c_{t,1}, \ldots, c_{t,k_t}), \qquad \bar{\mathbf{y}}_s^{\mathbf{c}_s} = \prod_{i\in[k_s]} \bar{y}_{s,i}^{c_{s,i}}.$$

Denote by $N^B(\pi_1; i_2, \mathbf{j}_2; \ldots; i_r, \mathbf{j}_r)$ the number of $(r-1)$-tuples $(\pi_2, \ldots, \pi_r)$ of B_n-partitions such that π_s $(2 \le s \le r)$ is of type $(i_s, \mathbf{j}_s)$ and $\pi_1, \pi_2, \ldots, \pi_r$ intersect minimally. In the notation of f in (2.16), we get

$$N^B(\pi_1; i_2, \mathbf{j}_2; \ldots; i_r, \mathbf{j}_r) = c\left[\mathbf{x}_1^{\mathbf{j}_1} z_1^{i_1}\right] \sum_{\substack{\mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s=\mathbf{j}_s\\ 2\le s\le r}} \left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s} z_s^{i_s}\right] F_r$$

where

$$c = \mathbf{j}_1! \cdot (2i_1)!! \prod_{2\le s\le r} (2k_s)!!^{-1},$$

$$F_r = \prod_{t\in[r]} \prod_{\alpha_t\in[k_t]} \prod_{\substack{Y_p\in\left\{z_p, y_{p,1}, \bar{y}_{p,1}, \ldots, y_{p,k_p}, \bar{y}_{p,k_p}\right\}\\ t+1\le p\le r}} f.$$

Now, let $N^B(\pi_1, k_2, \ldots, k_r)$ be the number of $(r-1)$-tuples $(\pi_2, \ldots, \pi_r)$ of B_n-partitions such that π_s contains exactly k_s block pairs and $\pi_1, \pi_2, \ldots, \pi_r$ intersect minimally. Then

$$N^B(\pi_1, k_2, \ldots, k_r) = \sum_{\substack{i_s\ge 0,\, j_{s,1},\ldots,j_{s,k_s}\ge 1\\ j_{s,1}+\cdots+j_{s,k_s}+i_s=n}} N^B(\pi_1; i_2, \mathbf{j}_2; \ldots; i_r, \mathbf{j}_r) \tag{2.17}$$

We claim that the condition $j_{s,1} + \cdots + j_{s,k_s} + i_s = n$ can be dropped in the above summation. In fact, the factor f in (2.16) contributes to x_1 or z_1 at most once with respect to the degree, and the contribution of f to x_1 or z_1 equals the contribution of f to $\mathbf{x}_s$, $\mathbf{y}_s$, $\bar{\mathbf{y}}_s$, or z_s, for any $2 \le s \le r$. Therefore the sum of the degrees of $\mathbf{x}_s$, $\mathbf{y}_s$, $\bar{\mathbf{y}}_s$, and z_s, equals the sum of the degrees of x_1 and z_1, that is, for any $2 \le s \le r$,

$$i_s + j_{s,1} + \cdots + j_{s,k_s} = i_1 + j_{1,1} + \cdots + j_{1,k_1} = n \tag{2.18}$$

Hence we can ignore the conditions (2.18) in (2.17). This implies that

$$N^B(\pi_1, k_2, \ldots, k_r) = c\left[\mathbf{x}_1^{\mathbf{j}_1} z_1^{i_1}\right] \sum_{i_s\ge 0,\, \mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s\ge \mathbf{1}} \left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s} z_s^{i_s}\right] F_r.$$

where $\mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s\geq \mathbf{1}$ indicates that $a_{s,h_s}+b_{s,h_s}+c_{s,h_s}\geq 1$ for any $1\leq h_s\leq k_s$. We will compute $\sum\left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s}z_s^{i_s}\right]F_r$ for $s=2,3,\ldots,r$ by the following procedure. First, for $s=2$, we have

$$\sum_{i_2\geq 0,\,\mathbf{a}_2+\mathbf{b}_2+\mathbf{c}_2\geq\mathbf{1}}\left[\mathbf{x}_2^{\mathbf{a}_2}\mathbf{y}_2^{\mathbf{b}_2}\bar{\mathbf{y}}_2^{\mathbf{c}_2}z_2^{i_2}\right]F_r=\sum_{l_2}\binom{k_2}{l_2}(-1)^{k_2-l_2}F_{r,2},$$

where

$$F_{r,2}=\prod_{\alpha_1,Y_p}(1+x_1^{\alpha_1}Y_3\cdots Y_r)^{2l_2+1}\prod_{Y_p}(1+z_1Y_3\cdots Y_r)^{l_2}\prod_{t\geq 3,\,\alpha_t,\,Y_p}(1+z_1z_3\cdots z_{t-1}x_t^{\alpha_t}Y_{t+1}\cdots Y_r).$$

So $N^B(\pi_1,k_2,\ldots,k_r)$ equals

$$c\left[\mathbf{x}_1^{\mathbf{j}_1}z_1^{i_1}\right]\sum_{l_2}\binom{k_2}{l_2}(-1)^{k_2-l_2}\sum_{\substack{i_s\geq 0,\,\mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s\geq\mathbf{1}\\ 3\leq s\leq r}}\left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s}z_s^{i_s}\right]F_{r,2}. \tag{2.19}$$

To compute the inner summation, let

$$g_s=\frac{1}{2}\left(\prod_{2\leq i\leq s}(2l_i+1)-1\right).$$

For any $s\geq 2$, it is clear that

$$(2l_{s+1}+1)g_s+l_{s+1}=g_{s+1}.$$

Starting with (2.19), we can continue the above procedure to deduce that for $2\leq h\leq r-1$,

$$N^B(\pi_1,k_2,\ldots,k_r)=c\left[\mathbf{x}_1^{\mathbf{j}_1}z_1^{i_1}\right]\sum_{l_2,\ldots,l_h}\prod_{2\leq i\leq h}\binom{k_i}{l_i}(-1)^{k_i-l_i}\sum_{\substack{i_s\geq 0,\,\mathbf{a}_s+\mathbf{b}_s+\mathbf{c}_s\geq\mathbf{1}\\ h+1\leq s\leq r}}\left[\mathbf{x}_s^{\mathbf{a}_s}\mathbf{y}_s^{\mathbf{b}_s}\bar{\mathbf{y}}_s^{\mathbf{c}_s}z_s^{i_s}\right]F_{r,h},$$

where

$$\begin{aligned}F_{r,h}=&\prod_{\alpha_1,Y_p}(1+x_1^{\alpha_1}Y_{h+1}\cdots Y_r)^{\prod_{2\leq i\leq h}(2l_i+1)}\prod_{Y_p}(1+z_1Y_{h+1}\cdots Y_r)^{g_h}\\ &\cdot\prod_{t\geq h+1,\,\alpha_t,\,Y_p}(1+z_1z_{h+1}\cdots z_{t-1}x_t^{\alpha_t}Y_{t+1}\cdots Y_r)\,.\end{aligned}$$

In particular, for $h=r-1$, we have

$$N^B(\pi_1,k_2,\ldots,k_r)=c\left[\mathbf{x}_1^{\mathbf{j}_1}z_1^{i_1}\right]\sum_{l_2,\ldots,l_{r-1}}\left(\prod_{2\leq i\leq r-1}\binom{k_i}{l_i}(-1)^{k_i-l_i}\right)G \tag{2.20}$$

where

$$G = \sum_{\mathbf{a}_r+\mathbf{b}_r+\mathbf{c}_r\geq \mathbf{1}} \left[\mathbf{x}_r^{\mathbf{a}_r}\mathbf{y}_r^{\mathbf{b}_r}\bar{\mathbf{y}}_r^{\mathbf{c}_r}\right] \prod_{\alpha_1, Y_p} (1+x_1^{\alpha_1})^{\prod_{2\leq i\leq r-1}(2l_i+1)} \prod_{Y_p} (1+z_1)^{g_{r-1}} \prod_{\alpha_r} (1+z_1 x_r^{\alpha_r})$$
$$= \sum_{l_r} \binom{k_r}{l_r} (-1)^{k_r-l_r} (1+z_1)^{g_r} \prod_{\alpha_1} (1+x_1^{\alpha_1})^{\prod_{2\leq i\leq r}(2l_i+1)}.$$

Since the number of B_n-partitions of type $\mathbf{j}_1$ equals

$$c' = \binom{n}{i_1}\binom{n-i_1}{\mathbf{j}_1}\frac{2^{n-i_1-k_1}}{k_1!} = \frac{(2n)!!}{(2i_1)!!(2k_1)!!\mathbf{j}_1!},$$

by (2.20), we obtain

$$N_{n,r}^B = \sum_{\substack{j_{1,1},\ldots,j_{1,k_1}\geq 1\\ i_1+j_{1,1}+\cdots+j_{1,k_1}=n}} c' \sum_{k_2,\ldots,k_r} N^B(\pi_1, k_2, \ldots, k_r)$$
$$= (2n)!! \sum_{\substack{k_2,\ldots,k_r\\ l_2,\ldots,l_r}} \left(\prod_{2\leq s\leq r} \binom{k_s}{l_s} \frac{(-1)^{k_s-l_s}}{(2k_s)!!}\right) \sum_{i_1,k_1} \frac{1}{(2k_1)!!} \left[z_1^{i_1}\right] (1+z_1)^{g_r} H \tag{2.21}$$

where

$$H = \sum_{\substack{i_1+j_{1,1}+\cdots+j_{1,k_1}=n\\ j_{1,1},\, j_{1,2},\, \ldots,\, j_{1,k_1}\geq 1}} \left[\mathbf{x}_1^{\mathbf{j}_1}\right] \prod_{\alpha_1} (1+x_1^{\alpha_1})^{\prod_{2\leq i\leq r}(2l_i+1)}$$
$$= \sum_{l_1} \binom{k_1}{l_1} (-1)^{k_1-l_1} \left[x^{n-i_1}\right] (1+x)^{l_1 \prod_{2\leq i\leq r}(2l_i+1)}.$$

Using the identity

$$\sum_k \binom{k}{l} \frac{(-1)^{k-l}}{(2k)!!} = \frac{e^{-1/2}}{(2l)!!}, \tag{2.22}$$

we can simplify the summation over $k_1, k_2, \ldots, k_r \geq 0$ in (2.21) to deduce that

$$N_{n,r}^B = (2n)!! \sum_{\substack{k_1,k_2,\ldots,k_r\\ l_1,l_2,\ldots,l_r}} \left(\prod_{t\in[r]} \binom{k_t}{l_t} \frac{(-1)^{k_t-l_t}}{(2k_t)!!}\right) \sum_{i_1} \left[x^{n-i_1} z_1^{i_1}\right] (1+z_1)^{g_r} (1+x)^{l_1 \prod_{2\leq i\leq r}(2l_i+1)}$$
$$= \frac{(2n)!!}{e^{r/2}} \sum_{l_1,l_2,\ldots,l_r} \frac{1}{(2l_1)!!(2l_2)!!\cdots(2l_r)!!} \left[x^n\right] (1+x)^{g_r + l_1 \prod_{2\leq i\leq r}(2l_i+1)}. \tag{2.23}$$

To further simplify the above summation, we observe that

$$g_r + l_1 \prod_{2\leq i\leq r} (2l_i+1) = \frac{1}{2}\left(\prod_{t\in[r]} (2l_t+1) - 1\right). \tag{2.24}$$

Substituting (2.24) into (2.23), we arrive at (2.14). This completes the proof. ∎

For example, we have $N_{1,r} = 2^r - 1$ and $N_{2,3}^B = 187$.

3 Minimally intersecting B_n-partitions without zero-block

In this section, we investigate the meet-semilattice of B_n-partitions without zero-block. Note that the minimal B_n-partition without zero-block is $\hat{0}^B$. Inspecting the proof of Theorem 2.1, we can restrict our attention to the set of B_n-partitions without zero-block by setting $i_0 = 0$ and $X = \emptyset$ in (2.8).

Corollary 3.1. *Let π be a B_n-partition consisting of k block pairs of sizes $i_1, i_2, \ldots, i_k$ listed in any order. For a given $l \geq 1$, the number $N^D(\pi; l)$ of B_n-partitions π' consisting of l block pairs, which intersects π minimally, is equal to*

$$N^D(\pi; l) = \frac{\mathbf{i}!}{(2l)!!} \left[\mathbf{x}^{\mathbf{i}}\right] \left(\prod_{\alpha \in [k]} (1 + x_\alpha)^2 - 1 \right)^l . \tag{3.1}$$

The total number of B_n-partitions without zero-block that intersect π minimally is given by

$$N^D(\pi) = \frac{\mathbf{i}!}{\sqrt{e}} \left[\mathbf{x}^{\mathbf{i}}\right] \exp\left(\frac{1}{2} \prod_{\alpha \in [k]} (1 + x_\alpha)^2 \right) . \tag{3.2}$$

For example, let $n = 3$, $\pi = \{\pm\{2\}, \pm\{1, -3\}\}$ and $l = 2$. Then (3.1) yields $N^D(\pi; 2) = 5$. In fact, the B_n-partitions consisting of 2 block pairs which intersect π minimally are exactly the 5 partitions consisting of two block pairs except for π itself.

Let N_n be the number of B_n-partitions without zero-block. Taking $\pi = \hat{0}^B$ in (3.2), we obtain the following formula.

Corollary 3.2. *We have*

$$N_n = \frac{1}{\sqrt{e}} \sum_{k \geq 0} \frac{(2k)^n}{(2k)!!}. \tag{3.3}$$

Let $N_n(k)$ denote the number of B_n-partitions containing k block pairs but no zero-block. It should be noted that the formula (3.3) can be easily deduced from the relation

$$N_n(k) = 2^{n-k} S(n, k),$$

where $S(n, k)$ are the Stirling numbers of the second kind, and the following identity on the Bell polynomials [9, 10]:

$$\sum_{k=0}^{n} S(n, k) x^k = \frac{1}{e^x} \sum_{k \geq 0} \frac{k^n}{k!} x^k.$$

The sequence $\{N_n\}_{n\geq 0}$ is A004211 in [12]:

$$1,\ 1,\ 3,\ 11,\ 49,\ 257,\ 1539,\ 10299,\ 75905,\ 609441,\ \ldots.$$

The proof of Corollary 2.4 implies the following corollary.

Corollary 3.3. *Let $N^D_{n,2}(k)$ denote the number of ordered pairs (π,π') of minimally intersecting B_n-partitions without zero-block such that π consists of exactly k block pairs. Then*

$$N^D_{n,2}(k) = \frac{(2n)!!}{(2k)!!\sqrt{e}}\,[x^n]\sum_{j\geq 0}\frac{1}{(2j)!!}\left[(1+x)^{2j}-1\right]^k.$$

The number $N^D_{n,2}$ of ordered pairs (π,π') of minimally intersecting B_n-partitions without zero-block is given by

$$N^D_{n,2} = \frac{2^n}{e}\sum_{k,\,l\geq 0}\frac{(2kl)_n}{(2k)!!\,(2l)!!}.$$

For example, $N^D_{1,2}=1$, $N^D_{2,2}=7$, $N^D_{3,2}=75$. The following theorem is an analogue of Theorem 2.6 with respect to the meet-semilattice of B_n-partitions without zero-block.

Theorem 3.4. *For $r\geq 2$, the number of minimally intersecting r-tuples $(\pi_1,\pi_2,\ldots,\pi_r)$ of B_n-partitions without zero-block equals*

$$N^D_{n,r} = \frac{2^n}{e^{r/2}}\sum_{k_1,k_2,\ldots,k_r}\frac{(2^{r-1}k_1k_2\cdots k_r)_n}{(2k_1)!!(2k_2)!!\cdots(2k_r)!!}. \tag{3.4}$$

Proof. Let $1\leq t\leq r$. Let $\mathbf{j}_t=(j_{t,1},j_{t,2},\ldots,j_{t,k_t})$ be a composition of n. Assume that π_t is of type $(0;\mathbf{j}_t)$. Let $N^D(\pi_1,\mathbf{j}_2,\ldots,\mathbf{j}_r)$ be the number of $(r-1)$-tuples $(\pi_2,\ldots,\pi_r)$ of such B_n-partitions such that $(\pi_1,\pi_2,\ldots,\pi_r)$ is minimally intersecting. By the argument in the proof of Theorem 2.1, we find

$$N^D(\pi_1,\mathbf{j}_2,\ldots,\mathbf{j}_r) = c\cdot\left[\mathbf{x}^{\mathbf{j}_1}\right]\sum_{\mathbf{b}_s+\mathbf{c}_s=\mathbf{j}_s}\left[\mathbf{y}_2^{\mathbf{b}_2}\bar{\mathbf{y}}_2^{\mathbf{c}_2}\cdots\mathbf{y}_r^{\mathbf{b}_r}\bar{\mathbf{y}}_r^{\mathbf{c}_r}\right]f(\mathbf{j}), \tag{3.5}$$

where

$$c = \mathbf{j}_1!\prod_{2\leq s\leq r}(2k_s)!!^{-1},$$

$$f(\mathbf{j}) = \prod_{\substack{\alpha\in[k_1]\\ Y_s\in\left\{y_{s,1},\,\bar{y}_{s,1},\,\ldots,\,y_{s,k_s},\,\bar{y}_{s,k_s}\right\}}}(1+x_\alpha Y_2Y_3\cdots Y_r).$$

Let $N^D(\pi_1, k_2, \ldots, k_r)$ be the number of $(r-1)$-tuples $(\pi_2, \ldots, \pi_r)$ of B_n-partitions such that π_s consists of k_s block pairs, and $\pi_1, \pi_2, \ldots, \pi_r$ are minimally intersecting. It follows from (3.5) that

$$\begin{aligned} N^D(\pi_1, k_2, \ldots, k_r) &= c \cdot \left[\mathbf{x}^{\mathbf{j}_1}\right] \sum_{\mathbf{b}_s+\mathbf{c}_s=\mathbf{j}_s\geq \mathbf{1}} \left[\mathbf{y}_2^{\mathbf{b}_2}\cdots\bar{\mathbf{y}}_r^{\mathbf{c}_r}\right] f(\mathbf{j}) \\ &= \mathbf{j}_1! \sum_{l_2,\ldots,l_r} \left(\left[\mathbf{x}^{\mathbf{j}_1}\right] \prod_{\alpha\in[k_1]} (1+x_\alpha)^{2^{r-1}l_2\cdots l_r} \right) \prod_{2\leq s\leq r} \binom{k_s}{l_s} \frac{(-1)^{k_s-l_s}}{(2k_s)!!}. \end{aligned}$$

Consequently,

$$\begin{aligned} N^D_{n,r} &= \sum_{k_1} \frac{1}{(2k_1)!!} \sum_{\substack{j_{1,1}+\cdots+j_{1,k_1}=n \\ j_{1,1},\ldots,j_{1,k_1}\geq 1}} \frac{2^n n!}{\mathbf{j}_1!} \sum_{k_2,\ldots,k_r} N^D(\pi_1, k_2, \ldots, k_r) \\ &= (2n)!! \sum_{\substack{k_1,k_2,\ldots,k_r \\ l_1,l_2,\ldots,l_r}} \prod_{1\leq s\leq r} \binom{k_s}{l_s} \frac{(-1)^{k_s-l_s}}{(2k_s)!!} [x^n](1+x)^{2^{r-1}l_1l_2\cdots l_r}. \end{aligned}$$

Applying (2.22), we can restate the above formula in the form of (3.4). This completes the proof. ∎

For example, when $n = 2$ and $r = 3$, by (3.4) we find that $N^D_{2,3} = 25$. In fact, there are 3 B_2-partitions without zero-block, that is,

$$0^B,\ \pi_1 = \{\pm\{1,2\}\},\ \pi_2 = \{\pm\{1,-2\}\}.$$

Among all $3^3 = 27$ 3-tuples of B_2-partitions without zero-block, only (π_1, π_1, π_1) and (π_2, π_2, π_2) are not minimally intersecting.

Corollary 3.5. *We have*

$$N^D_{n,r} = \sum_{j=1}^{n} N^r_j 2^{n-j} s(n,j), \tag{3.6}$$

where $s(n,j)$ are the Stirling numbers of the first kind. Moreover,

$$M^D_r\left(\frac{e^{2x}-1}{2}\right) = \sum_{n\geq 0} N^r_n \frac{x^n}{n!}, \tag{3.7}$$

where

$$M^D_r(x) = \sum_{n\geq 0} N^D_{n,r} \frac{x^n}{n!}.$$

The formula (3.6) can be considered as a type B analogue of Wilf's formula (1.2), whereas (3.7) is analogous to Canfield's formula (1.3).

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